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Another view of the Gell-Mann-Okubo mass formula

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The Gell-Mann-Okubo mass formula for light hadrons assumes that mass effective operator is in the 8 representation of flavour SU(3). To be in very good agreement with experimental data, we have to assume that the mass effective operator is in part also in the 27 representation. From that, we derive improved mass formulas.

Fifty five years ago, the mass formula of Gell-Mann-Okubo (GMO) for light hadrons (composed of valence quarks u, d, s) met a great success in the elucidation of their mass spectrum in terms of flavour SU(3) singlets, octets and decuplet.[2]

In this note, we propose to review GMO mass formula and to improve it.

We start with GMO for the baryon octet[3], expressed as

$$2M_N + 2M_\Xi = 3M_\Lambda + M_\Sigma \quad (1)$$

with[4]

$$M_N \equiv [m(p) + m(n)]/2 \approx 938.9 \text{ MeV} \quad (2a)$$

$$M_\Xi \equiv [m(\Xi^\circ) + m(\Xi^-)]/2 \approx 1318.3 \text{ MeV} \quad (2b)$$

$$M_\Lambda \equiv m(\Lambda) \approx 1115.7 \text{ MeV} \quad (2c)$$

$$M_\Sigma \equiv [m(\Sigma^-) + m((\Sigma^\circ) + m(\Sigma^+)]/3 \approx 1193.2 \text{ MeV} \quad (2d)$$

In fact,

$$2M_N + 2M_\Xi - 3M_\Lambda - M_\Sigma \approx -25.8 \text{ MeV} \quad (3)$$

GMO is satisfied up to 1 % because

$$25.8 \text{ MeV} / 2M_\Xi \approx 0.01$$

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2 M. Gell-Mann, unpublished, http://www.osti.gov/accomplishments/documents/fullText/ACC0113.pdf, 25 march 1961; S.Okubo, Progr. Theor. Phys. (Kyoto), **27**, 949 (1962); M. Gell-Mann and Y Ne'eman, *The Eightfold Way*, W.A. Benjamin,Inc., New-York (1964).

3 We performed a similar analysis for the baryon decuplet. Here we have to take in consideration 8, 27 and 64 representations of flavour SU(3) as components of mass effective operator.

4 Hadron experimental values used in this note come from: C.Patrigani et al. (Particle Data Group), Chin. Phys. C, **40**, 100001 (2016), http://pdg.lbl.gov/2016/tables/contents_tables.html.

Now, let us look GMO mass formula from an other point of view, expressing it as

$$2M_N + 2M_\Xi - 4M_\Lambda = M_\Sigma - M_\Lambda \qquad (4)$$

As indicated in eq. (3), this formula is approximated and is quite different of the expression

$$3M_N + 3M_\Xi - 6M_\Lambda = M_\Sigma - M_\Lambda \qquad (5a)$$

which is in excellent agreement with experimental data. Let us note that eq. (5a) is equivalent to

$$2M_N + 2M_\Xi - 3M_\Lambda - M_\Sigma = -[M_\Sigma - M_\Lambda]/3 \approx -25.8 \text{ MeV} \qquad (5b)$$

This is the GMO improved mass formula for baryons.

We are going to show from where comes eqs. (5).

To derive GMO mass formula, we use the interaction picture of the perturbation theory at the first order, with state vectors (in the limit where flavour SU(3) is exact) sandwiching mass effective operator which breaks flavour SU(3) in the direction of the hypercharge axis Y. This mass effective operator has two components, one belonging to 8 representation and another one belonging to 27 representation of flavour SU(3).

Mass of baryon B, belonging to light octet representation is expressed then as

$$M_B = M_0 - \Delta_{8a}Y/2 + (\Delta_{8s} + \Delta_{27})[I(I + 1) - Y^2/4]/10 + \Delta_{27}Y^2/4 \qquad (6)$$

or, explicitly as

$$M_N = M_0 - \Delta_{8a}/2 + (\Delta_{8s} + \Delta_{27})/20 + \Delta_{27}/4 \qquad (6a)$$

$$M_\Xi = M_0 + \Delta_{8a}/2 + (\Delta_{8s} + \Delta_{27})/20 + \Delta_{27}/4 \qquad (6b)$$

$$M_\Lambda = M_0 \qquad (6c)$$

$$M_\Sigma = M_0 + (\Delta_{8s} + \Delta_{27})/5 \qquad (6d)$$

M_0 is the mass SU(3) symmetric part. Δ_{8a}, Δ_{8s} et Δ_{27} are contributions to baryon octet masses corresponding to antisymmetric 8, symmetric 8 and 27 representations of flavour SU(3) group[5] respectively:

$$\Delta_{8a} = M_\Xi - M_N \approx 379.4 \text{ MeV} \qquad (7)$$

$$\Delta_{8s} = -2M_N - 2M_\Xi - 2M_\Lambda + 6M_\Sigma \approx 413.1 \text{ MeV} \qquad (8)$$

$$\Delta_{27} = 2M_N + 2M_\Xi - 3M_\Lambda - M_\Sigma \approx -25.8 \text{ MeV} \qquad (9)$$

GMO mass formula [eqs. (1) or (4)] is obtained assuming octet dominance in the mass effective operator, i.e. assuming

$$\Delta_{27} = 0 \qquad (10)$$

5 For definitions of Δ_{8a}, Δ_{8s} et Δ_{27} , see table 11 (p. 26) of W. Bieterholz et al., http://arxiv.org/pdf/1102.5300v2.pdf, 2011.

In contradistinction, if we wish to satisfy eqs. (5) a mass formula in strict agreement with experiments, we find from eqs. (6) that necessarily

$$(\Delta_{8s} + \Delta_{27})/5 = -\,3\,\Delta_{27} \qquad (11)$$

and that

$$\Delta_{27} = -\,[\,M_\Sigma - M_\Lambda]\,/3 \approx -\,25.8 \text{ MeV} \qquad (12)$$

It is obvious that eqs. (5) are equivalent to

$$3M_N + 3M_\Xi = 5M_\Lambda + M_\Sigma \qquad (13)$$

to be compared to eq. (1). Eq. (1) is not an approximated result due to flavour SU(3) breaking but to the neglect of 27 representation in the mass effective operator. Eqs. (5) or (13) are in agreement with experiments at least at the same level of precision as the Coleman-Glashow mass formula.[6]

$$m(n) - m(p) + m(\Xi^-) - m((\Xi^\circ) = m(\Sigma^-) - m(\Sigma^+) \qquad (14)$$

Next, we extend our results to four light mesons octets with $J^{PC} = 0^{-+}$, 1^{--}, 2^{++} and 3^{--}, denominated J = 0, 1, 2 and 3 respectively.

We assume that mass formula for light mesons will be mass quadratic.

For mesons , we have a mixing between singlet state under flavour SU(3) and the I = 0 member of the corresponding octet: $\eta - \eta'$ (0^{-+}), $\omega - \varphi$ (1^{--}), $f_2 - f_2'$ (2^{++}), $\omega_3 - \varphi_3$ (3^{--}).

For each octet, we have a mixing angle denoted θ_J.

The GMO mass formulas for light mesons[7] are expressed as

$$4M_K^2 - 3M_{\eta 8}^2 - M_\pi^2 = 0, \qquad \theta_0 \approx -\,11.4° \qquad (15a)$$

$$4M_{K^*}^2 - 3M_{\omega 8}^2 - M_\rho^2 = 0, \qquad \theta_1 \approx 39.1° \qquad (15b)$$

$$4M_{K^*2}^2 - 3M_{f\,2\_8}^2 - M_{a2}^2 = 0, \qquad \theta_2 \approx 32.1° \qquad (15c)$$

$$4M_{K^*3}^2 - 3M_{\omega 3\_8}^2 - M_{\rho 3}^2 = 0, \qquad \theta_3 \approx 31.8° \qquad (15d)$$

We propose an alternative to GMO mass formulas given in eqs. (15 a-d):

$$4M_K^2 - 3M_{\eta 8}^2 - M_\pi^2 = +\,[\,M_\pi^2 - M_{\eta 8}^2]/3, \qquad \theta_0 \approx -\,18° \qquad (16a)$$

$$4M_{K^*}^2 - 3M_{\omega 8}^2 - M_\rho^2 = +\,[\,M_\rho^2 - M_{\omega 8}^2]/3, \qquad \theta_1 \approx 35.2° \qquad (16b)$$

$$4M_{K^*2}^2 - 3M_{f\,2\_8}^2 - M_{a2}^2 = -\,(M_{a2}^2 - M_{f\,2\_8}^2)/3, \qquad \theta_2 \approx 35.2° \qquad (16c)$$

$$4M_{K^*3}^2 - 3M_{\omega 3\_8}^2 - M_{\rho 3}^2 = -\,(M_{\rho 3}^2 - M_{\omega 3\_8}^2)/3, \qquad \theta_3 \approx 35.2° \qquad (16d)$$

with $\sin\theta_0 = -\,1/(\sqrt{5} + 1)$ and $\sin\theta_J = 1/\sqrt{3}$ when J = 1, 2 or 3.

$\sin\theta_J = 1/\sqrt{3}$ separates I = 0 meson states containing (ssbar) only and the ones containing (uubar + ddbar) only as valence quarks.

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6 S.Coleman and S.L. Glashow, Phys.Rev. Lett., **6**, 423 (1961). See also P.A. Carruthers, *Introduction to unitary symmetry*, Interscience Publishers (1966).

7 http://pdg.lbl.gov/2016/reviews/rpp2016-rev-quark-model.pdf, table 15.2.

$\sin\theta_0 = -1/(\sqrt{5}+1)$ [8] is compatible with different η − η' mixing determinations :
$-14° > \theta_0 > -22°$, taking into account axial strong anomaly.

To derive GMO mass formulas [eqs. (15a-d)] and improved GMO mass formulas [eqs. 16a-d)], we proceed as for baryon octet.

Masses of mesons of isospin I, belonging to light octet representation J, are expressed[9] as

$$M^2_{J,I} = \mathit{M}^2_{J,0} + (\Delta'_{J,8} + \Delta'_{J,27})[I(I+1) - Y^2/4]/10 + \Delta'_{J,27}Y^2/4 \qquad (17)$$

or, explicitly as

$$M^2_{J,1/2} = \mathit{M}^2_{J,0} + (\Delta'_{J,8} + \Delta'_{J,27})/20 + \Delta'_{J,27}/4 \qquad (17a)$$

$$M^2_{J,0} = \mathit{M}^2_{J,0} \qquad (17b)$$

$$M^2_{J,1} = \mathit{M}^2_{J,0} + (\Delta'_{J,8} + \Delta'_{J,27})/5 \qquad (17c)$$

$\mathit{M}_{J,0}$ is the mass SU(3) symmetric part. $\Delta'_{J,8}$ et $\Delta'_{J,27}$ are contributions to meson octet (masses)² corresponding to 8 and 27 representations of flavour SU(3) group[10] respectively:

$$\Delta'_{J,8} = -4M^2_{J,1/2} - 2M^2_{J,0} + 6M^2_{J,1} \qquad (18)$$

$$\Delta'_{J,27} = 4M^2_{J,1/2} - 3M^2_{J,0} - M^2_{J,1} \qquad (19)$$

GMO mass formulas [eqs. (15a-d)] are obtained assuming octet dominance in the (mass)² effective operator, i.e. assuming

$$\Delta'_{J,27} = 0 \qquad (20)$$

Modified and improved GMO mass formulas [eqs. 16a-d] are obtained assuming

$$(\Delta'_{J,8} + \Delta'_{J,27})/5 = +3\,\Delta'_{J,27} \qquad (21)$$

when J = 0 or 1

and

$$(\Delta'_{J,8} + \Delta'_{J,27})/5 = -3\,\Delta'_{J,27} \qquad (22)$$

when J = 2 or 3

Eqs. (20), (21) and (22) imply that, to be in agreement with experimental data, (mass)² effective operator for light hadrons belonging to octet must be part not only of 8 representation but also of 27 representation. Furthermore, there is an intimate relation between contributions coming from 8 and 27 representations.

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8 It is amusing to note that $(\sqrt{5}+1)/2$ is the golden ratio.

9 A meson of isospin I in an octet has one possible value for Y^2. I = ½, 0 and 1 imply Y^2 = 1, 0, 0 respectively.

10 For definitions of $\Delta'_{J,8}$ et $\Delta'_{J,27}$, see table 12 (p. 26) of W. Bieterholz et al., http://arxiv.org/pdf/1102.5300v2.pdf, 2011.

Appendix

From eq. (16a), we get

$M_{\eta 8} \approx 599$ MeV

in agreement with the value obtained from relation

$M_{\eta 8}^2 = M_{\eta}^2 \cos^2\theta_0 + M_{\eta'}^2 \sin^2\theta_0$ ($\theta_0 = -18°$)

From eq. (16b), we get

$M_{\omega 8} \approx 947$ MeV

in agreement with the value obtained from relation

$M_{\omega 8}^2 = M_{\omega}^2 \sin^2\theta_1 + M_{\varphi}^2 \cos^2\theta_1$ ($\theta_1 = 35.2°$)

From eq. (16c), we get

$M_{f2\_8} \approx 1450$ MeV

in agreement with the value obtained from relation

$M_{f2\_8}^2 = M_{f2}^2 \sin^2\theta_2 + M_{f'2}^2 \cos^2\theta_2$ ($\theta_2 = 35.2°$)

From eq. (16d), we get

$M_{\omega 3\_8} \approx 1793$ MeV

in agreement with the value obtained from relation

$M_{\omega 3\_8}^2 = M_{\omega 3}^2 \sin^2\theta_3 + M_{\varphi 3}^2 \cos^2\theta_3$ ($\theta_3 = 35.2°$)